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Extension of maps into nilpotent spaces III

M. Cencelj [a,1], A.N. Dranishnikov [b,*,2]

[a] IMFM, University of Ljubljana, POB 2964, SI-1001 Ljubljana, Slovenia
[b] University of Florida, Department of Mathematics, PO Box 118105, 358 Little Hall, Gainesville, FL 32611-8105, USA

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Abstract

Let M be a nilpotent CW-complex. We give necessary and sufficient cohomological dimension theory conditions for a finite-dimensional metric compactum X so that every map $A \to M$, where A is a closed subset of X, can be extended to a map $X \to M$.

This is a generalization of a result by Dranishnikov [Mat. Sb. (1991)] where such conditions were found for simply-connected CW-complexes M, and Cencelj and Dranishnikov [Canad. Bull. Math. (2001) and Topology Appl. (2002)] where a condition of finitely generatedness was imposed on the nilpotent CW-complex M.

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We generalize the main theorem of [3] and Theorem 7 of [1] to obtain the following theorem. We use the Kuratowski notation $X\tau M$ for the case every map from a closed subset of X to M can be extended over all of X. We recall that the cohomological dimension

* Corresponding author.
E-mail addresses: matija.cencelj@uni-lj.si (M. Cencelj), dranish@math.ufl.edu (A.N. Dranishnikov).
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of a space X can be defined in these notations as follows: $\dim_G X \leqslant n$ if and only if $X \tau K(G,n)$, where $K(G,n)$ is an Eilenberg–MacLane complex. The nth symmetric power $SP^n M$ of a topological space M is the orbit space of the natural action of the nth symmetric group S_n on the nth power M^n. For a pointed space M there is a natural inclusion $SP^n M \subset SP^{n+1} M$ which gives rise to the direct system. The infinite symmetric power of a space $SP^\infty M$ is the direct limit of this system.

Theorem 1. *For any nilpotent* CW*-complex M and finite-dimensional metric compactum X, the following are equivalent*:

(1) $X \tau M$;
(2) $X \tau SP^\infty M$;
(3) $\dim_{H_i(M)} X \leqslant i$ *for every* $i > 0$;
(4) $\dim_{\pi_i(M)} X \leqslant i$ *for every* $i > 0$.

We recall that a group G is called *nilpotent* if its lower central series $G = \Gamma^1 G \supset \Gamma^2 G \supset \cdots \supset \Gamma^k G \subset \Gamma^{k+1} G = 1$ has a finite length k called the nilpotency class of G. Here $\Gamma^2 G = [G,G]$ and $\Gamma^i G = [G, \Gamma^{i-1} G]$. The main examples of nilpotent groups are upper triangular matrix groups. An action $\alpha : G \to \operatorname{Aut} H$ is called nilpotent if there is a G-invariant normal stratification $H = H_1 \supset \cdots \supset H_i \supset \cdots \supset H_n = *$ such that H_i / H_{i+1} is Abelian and the induced action on H_i / H_{i+1} is trivial for all i. A topological space is called *nilpotent* [5] if the action of the fundamental group $\pi_1(X)$ on every higher dimensional homotopy group is nilpotent. In particular, it implies that $\pi_1(X)$ is nilpotent.

We recall that for every Abelian group G there exists a Bockstein family $\sigma(G)$ of Abelian groups [3] such that for every metric compactum X we have

$$\dim_G X = \max_{H \in \sigma(G)} \dim_H X.$$

The family $\sigma(G)$ is a subfamily of the family $\sigma = \mathbb{Q} \cup (\bigcup_p \sigma_p)$, where $\sigma_p = \{\mathbb{Z}_p, \mathbb{Z}_{p^\infty}, \mathbb{Z}_{(p)}\}$. Here $\mathbb{Z}_{p^\infty}$ is the direct limit of the groups $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_{(p)} = \{m/n;\ n \text{ not divisible by } p\}$ is the p-localization of the integers. The family $\sigma(G)$ is defined by the following rule: $\mathbb{Z}_{(p)} \in \sigma(G)$ if and only if $F(G)$ is not p-divisible; $\mathbb{Z}_p \in \sigma(G)$ if and only if the group G_p is not p-divisible; $\mathbb{Z}_{p^\infty} \in \sigma(G)$ if and only if $G_p \neq 0$ and G_p is p-divisible; and $\mathbb{Q} \in \sigma(G)$ if $F(G) \neq 0$. Here G_p is the p-torsion subgroup of G and $F(G) = G/\operatorname{Tor}(G)$.

We refer the reader to [2] and [4] for the facts and terminology from the cohomological dimension theory which we are using in this paper.

As opposed to Theorem 7 of [1] and Theorem 1 of [2] we do not impose any additional condition on M except nilpotency. The proof of the main theorem, however, is not dimension-wise in all cases, in some cases dimension over $H_1(M)$ does not provide as much information as dimension over $\pi_1(M)$, however, dimension over $H_1(M)$ and over $H_2(M)$ together suffice.

Theorem 2. *For a nilpotent group N and every metric compactum X the following equivalence holds*

$$\dim_N X \leqslant 1 \iff \dim_{\mathrm{Ab}\, N} X \leqslant 1.$$

Provided N has one of the following properties:

(1) *N is a torsion group*;
(2) *for every prime p for which $p - \operatorname{tor} N \neq 1$*
  (a) *N is not p-divisible, or*
  (b) *$p - \operatorname{tor}(\operatorname{Ab} N) \neq 0$.*

Proof. We need to prove

$$\dim_{\operatorname{Ab} N} X \leqslant 1 \implies \dim_N X \leqslant 1$$

for every metric compactum X, the reverse implication holds for every group.

Recall some general results which we will use in the sequel. A nilpotent group is the result of finitely many central extensions

$$0 \to A_i \to N_{i+1} \to N_i \to 1$$

of an Abelian group N_1 (note that in this case $\dim_{A_i} X \leqslant 1$ and $\dim_{N_i} X \leqslant 1$ implies $\dim_{N_{i+1}} X \leqslant 1$).

At such an extension we have a natural epimorphism

$$\operatorname{Ab} N_i \to \operatorname{Ab} N_{i+1}. \tag{1}$$

For every group G we also have an epimorphism

$$\bigotimes^n \operatorname{Ab} G \to \Gamma^n(G)/\Gamma^{n+1}G, \quad x_1 \otimes \cdots \otimes x_n \mapsto \big[\ldots\big[[x_1, x_2], x_3\big], \ldots, x_n\big].$$

If the nilpotent group N is of nilpotency class n we therefore have an epimorphism

$$\bigotimes^n \operatorname{Ab} N \to \Gamma^n = A_n. \tag{2}$$

(I) If N is a torsion nilpotent group it is the direct sum of p-torsion nilpotent groups, p prime. Obviously every p-torsion nilpotent group is the result of finitely many central extensions of a p-torsion Abelian group by p-torsion Abelian groups.

If the p-torsion nilpotent group N is also p-divisible (radicable) we see from the epimorphisms (1) and (2) that every group in the central extensions has to be p-divisible. The same holds for $\operatorname{Ab} N$. Therefore $\{\mathbb{Z}_{p^\infty}\} = \sigma(\operatorname{Ab} N)$ (and similarly for all groups in the central extensions in the construction of N), from $\dim_{\operatorname{Ab} N} X \leqslant 1$ we obtain in finitely many steps $\dim_N X \leqslant 1$.

If the p-torsion nilpotent group N is not p-divisible the same holds for $\operatorname{Ab} N$ and therefore $\{\mathbb{Z}_p\} = \sigma(\operatorname{Ab} N)$. All groups in the central extensions in the construction of N have either $\mathbb{Z}_p$ or $\mathbb{Z}_{p^\infty}$ in their Bockstein family. Since $\dim_{\mathbb{Z}_p} X \leqslant 1$ implies also $\dim_{\mathbb{Z}_{p^\infty}} X \leqslant 1$ we obtain from $\dim_{\operatorname{Ab} N} X \leqslant 1$ also in this case $\dim_N X \leqslant 1$ in finitely many steps.

(II) If N is not torsion the same holds for $\operatorname{Ab} N$ and therefore $\dim_{\operatorname{Ab} N} X \leqslant 1$ implies $\dim_{\mathbb{Q}} X \leqslant 1$.

The prime numbers p for which $\mathbb{Z}_{(p)} \in \sigma(\operatorname{Ab} N)$ we have no problems (i.e., any of $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$ or $\mathbb{Z}_{p^\infty}$ can appear in the Bockstein family σ of any Abelian group in the central extensions of N).

Consider those prime p for which $\mathbb{Z}_{(p)} \notin \sigma(\operatorname{Ab} N)$, $\mathbb{Z}_p \in \sigma(\operatorname{Ab} N)$. Let N be the central extension

$$0 \to \Gamma^n \to N \to M \to 1.$$

The surjection $N \to M$ implies the surjection $\operatorname{Ab} N \to \operatorname{Ab} M$, therefore also the surjection $\operatorname{Ab} N/\operatorname{Tor} \to \operatorname{Ab} M/\operatorname{Tor}$. Therefore $\mathbb{Z}_{(p)} \notin \sigma(\operatorname{Ab} M)$.

The surjection

$$\bigotimes^n \operatorname{Ab} N \to \Gamma^n$$

and the surjection $\bigotimes^n \operatorname{Ab}(N/\operatorname{Tor}) \cong \bigotimes^n \operatorname{Ab} N/\operatorname{Tor} \to \Gamma^n$ imply $\mathbb{Z}_{(p)} \notin \sigma(\Gamma^n)$. By induction we see that for such a prime p and every Abelian group A appearing in the central extensions in the construction of N we have $\mathbb{Z}_{(p)} \notin \sigma(A)$.

Consider the prime numbers p for which we only have $\mathbb{Z}_{p^\infty} \in \sigma(\operatorname{Ab} N)$. This means that $\operatorname{Ab} N$ as well as N are p-divisible groups with p-torsion (this must hold also for N; if N is without p-torsion it is $\bar{p}$-local, $\bar{p}$ denotes the complement of $\{p\}$ in the set of all primes, implying that also H_1 is $\bar{p}$-local and in particular without p-torsion). In this case we see that no Abelian group A in the central extensions of N has either $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$ in its Bockstein family.

If the prime p does not appear in $\sigma(\operatorname{Ab} N)$ this group is p-divisible and without p-torsion. Then also N is p-divisible and by assumption it has no p-torsion. By induction we see that also all groups A in the central extensions of N are p-divisible and without p-torsion. □

Proof of Theorem 1. To start the proof (i.e., in dimension 1) we need only check those nilpotent groups $G = \pi_1(M)$ which are not torsion and for some prime p the abelianization $\operatorname{Ab} G$ has no p-torsion and is p-divisible while G has p-torsion (and is also p-divisible), see, e.g., [7, Example 5.2]. For all other prime numbers p the proof follows the proof of Theorem 2.

Since G_1 is a quotient of $\operatorname{Ab} G$ also G_1 must be p-divisible. If G_1 had p-torsion one of the A_i would not be divisible by p, but this is impossible since the p-divisibility of G implies the p-divisibility of all G_i and $\operatorname{Ab} G_i$. Therefore G_1 has no p-torsion and is therefore $\bar{p}$-local. The same holds for its homology $H_*(G_1)$. We can construct G from G_1 only with central extensions by p-divisible Abelian groups due to the epimorphisms (2). If we extend G_1 (or any other $\bar{p}$-local G_i) by an Abelian group without p-torsion the resulting group is $\bar{p}$-local. Therefore in this case we extend the $\bar{p}$-local group G_i at least once by an Abelian group A which has p-divisible p-torsion. Since the p-torsion does not appear in the abelianization of the extension it has to be eliminated by the boundary homomorphism

$$\partial : E^2_{2,0} = H_2(G_i) \to A = E^2_{0,1}$$

of the Lyndon–Hochschild–Serre spectral sequence (see, e.g., [6]) the kernel of which is a quotient of $H_2(G_{i+1})$. Since $H_2(G_i)$ is a $\bar{p}$-local group the p-torsion of A can be eliminated only if the kernel of ∂ contains an element of infinite order which is not p-divisible. Therefore $\mathbb{Z}_{(p)} \in \sigma H_2(G_{i+1})$ and thus $\mathbb{Z}_{(p)} \in \sigma H_2(G)$. Since there is an epimorphism

$$H_2(M) \to H_2\big(K\big(\pi_1(M), 1\big)\big) = H_2\big(\pi_1(M)\big),$$

we have also $\mathbb{Z}_{(p)} \in \sigma H_2(M)$.

Since by assumption $\mathrm{Ab}(\pi_1(M)) = H_1(M)$ is not torsion, $\dim_{H_1(M)} X \leqslant 1$ implies $\dim_{\mathbb{Q}} X \leqslant 1$. Therefore if X is p-regular we have

$$\dim_{\mathbb{Q}} X = \dim_{\mathbb{Z}_{p^\infty}} X = \dim_{\mathbb{Z}_p} X = \dim_{\mathbb{Z}_{(p)}} X \leqslant 1.$$

If, however, X is p-singular, $\dim_{H_1(M)} X \leqslant 1$ and $\dim_{H_2(M)} X \leqslant 1$ imply $\dim_{\mathbb{Z}_{p^\infty}} X \leqslant 1$, but this on the other hand implies (since this holds for all such p) that in finitely many steps we obtain $\dim_{\pi_1(M)} X \leqslant 1$.

The rest of the proof is essentially the same as for $\pi_1(M)$ finitely generated [2]. $\square$

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